B Homology of 4-manifolds

lemma 5:

If $X^4 = (0\text{-handle}) \cup k(2\text{-handles})$ then

$H_0(X) \cong \mathbb{Z}$, $H_1(X) = 0$ and

1) $H_2(X) \cong \oplus_k \mathbb{Z}$ generated by 2-handles

<u>specifically:</u> 2-handles attached to $L_1 \cup \ldots \cup L_k$, let Σ_i be a Seifert surface for L_k pushed into interior(B^4) and $A_i = \Sigma_i \cup C_i$ where C_i is the core of 2-handle

2) $H_2(X, \partial X) \cong \oplus_k \mathbb{Z}$ generated by the co-cores of the 2-handles

Proof:

1) let $X_i = (0\text{-handles}) \cup 1^{st}\ i\ (2\text{-handles})$

note $X_{i+1}/X_i \simeq S^2$

so
$$\underset{\substack{\shortparallel \\ 0}}{H_3(X_{i+1}, X_i)} \to H_2(X_i) \to H_2(X_{i+1}) \to \underset{\substack{\text{\tiny IS} \\ \pm 1 \in \mathbb{Z}}}{H_2(X_i, X_{i+1})} \to \underset{\substack{\shortparallel \\ 0}}{H_1(X_i)}$$
$$A_{i+1} \longmapsto \pm 1 \in \mathbb{Z}$$

$\therefore H_2(X_{i+1}) \cong H_2(X_i) \oplus \underset{\text{gen by } A_{i+1}}{\mathbb{Z}}$

so inductively $H_2(X)$ is as claimed.

2) since $H_1(X)=0$, Universal Coefficients gives

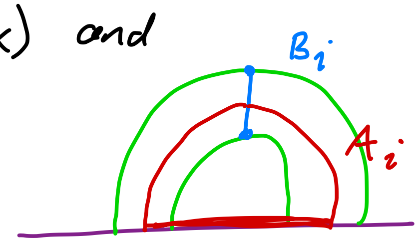
$$H^2(X) \cong H_2(X)$$

and Poincaré duality gives $H_2(X,\partial X) \cong H^2(X) \cong \oplus_k \mathbb{Z}$

note: if B_i is the cocore to i^{th} 2-handle

then $[B_i] \in H_2(X,\partial X)$ and

$$B_i \cdot A_j = \delta_{ij}$$



so B_i gives elt $H^2(X) \cong \mathrm{Hom}(H_2(X),\mathbb{Z})$

that is dual to A_i

i.e. B_i's generate $H_2(X,\partial X)$

Theorem 6:

let $L_1 \cup \ldots \cup L_k$ be a link in S^3 and X be the 4-mfd obtained from B^4 by attaching 2-handles to B^4 along the L_i with framing n_i

let $a_{ij} = \begin{cases} lk(L_i,L_j) & i \neq j \\ n_i & i=j \end{cases}$

the matrix $A=(a_{ij})$ is called the <u>linking matrix</u>

we have

$$\begin{array}{ccccccc} H_2(X) & \longrightarrow & H_2(X,\partial X) & \xrightarrow{\partial} & H_1(\partial X) & \longrightarrow & H_1(X) \\ \wr\wr & & \wr\wr & & & & \parallel \\ \oplus_k \mathbb{Z} & \xrightarrow{A} & \oplus_k \mathbb{Z} & & & & 0 \end{array}$$

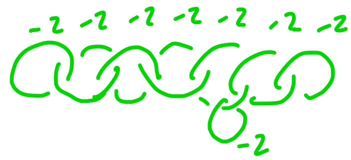
where we use A_i as a basis for $H_2(X)$ and

B_i " " $H_2(X,\partial X)$

Thus if μ_i is the meridian to L_i then we have a presentation for the homology of ∂X:

$$H_1(\partial X) \cong \langle \mu_1, \dots, \mu_n \mid a_{11}\mu_1 + a_{12}\mu_2 + \dots, \dots \rangle$$

example:

i) 

$$\text{linking matrix } A = \begin{pmatrix} -2 & 1 & & & & & & \\ 1 & -2 & 1 & & & & 0 & \\ & 1 & -2 & 1 & & & & \\ & & 1 & -2 & 1 & & & \\ & & & 1 & -2 & 1 & 0 & 1 \\ & 0 & & & 1 & -2 & 1 & 0 \\ & & & & 0 & 1 & -2 & 0 \\ & & & & 1 & 0 & 0 & -2 \end{pmatrix}$$ E_8 matrix

you can compute $\det A = 1$

so A is an isomorphism $\oplus_k \mathbb{Z} \to \oplus_k \mathbb{Z}$

and so $H_1(\partial X) = 1$

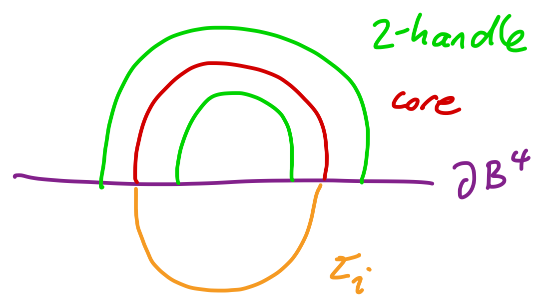
Proof: we first note that

$$H_2(X, \partial X) \longrightarrow H_1(\partial X)$$

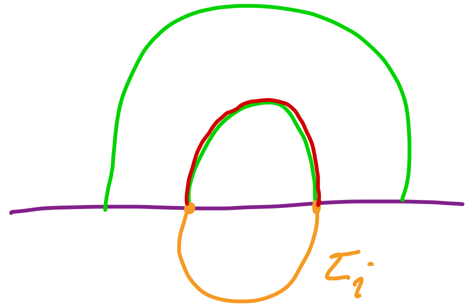
is just a boundary map so $\partial(B_i) = \mu_i$, so μ_i generate $H_1(\partial X)$

now $H_2(X) \to H_2(X, \partial X)$ is just inclusion so we need to write the A_i in terms of the B_i

recall $A_i = \Sigma_i \cup$ core

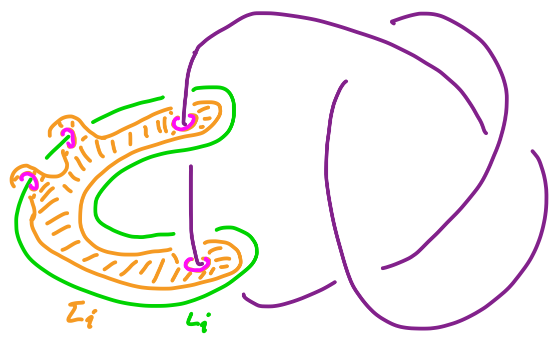


push core to ∂X



anything in ∂X "disapears" in $H_2(X, \partial X)$

so we just have Σ_i with its boundary linking L_i, n_i times in S^3



now push Σ_i into ∂X

can do this <u>except</u> for parts where some other component $L_j \cap \Sigma_i$ and where $L_i \cap \Sigma_i$

these are precisely meridian curves bounding B_i in X

so $A_i \longmapsto a_{1i} B_1 + a_{2i} B_2 + \dots + a_{ki} B_k$

We now study $H_2(X)$ when X is closed

Recall $H_2(X) \cong H^2(X)$ (if $H_1(X)$ has no torsion)

and $H^2(X)$ has a cup-product pairing

$$H^2(X) \times H^2(X) \longrightarrow H^4(X) \cong \mathbb{Z}$$

we interperate this geometrically

lemma 7:

> if X is a closed oriented 4-manifold then any $a \in H_2(X)$
> is represented by an oriented surface $\Sigma \subset X$
>
> i.e. $[\Sigma] = a$

Proof: $H_2(X) \cong H^2(X) \underset{\text{Brown}}{\cong} [X; K(\mathbb{Z},2)]$

now $K(\mathbb{Z},2) = \mathbb{C}P^\infty = \text{0-cell} \cup \text{2-cell} \cup \text{4-cell} \cup \ldots$

so any map $f: X \to \mathbb{C}P^\infty$ is homotopic to

$$f: X \to \mathbb{C}P^2$$

(indeed, $f(X) \subset \mathbb{C}P^n$ some n since it is compact

now make f transverse to center of $2n$-cell
thus f disjoint from it if $n > 2$, and so f can be homotoped of of it, i.e. into $\mathbb{C}P^{n-1}$

so inductively get $f(X) \subset \mathbb{C}P^2$)

$H_2(\mathbb{C}P^2) \cong \mathbb{Z}$ generated by $\mathbb{C}P^1 \subset \mathbb{C}P^2$

make f transverse to $\mathbb{C}P^1$ and set $\Sigma = f^{-1}(\mathbb{C}P^1)$

$$\begin{tikzcd}
H_2(X) \arrow[r, phantom, "\cong"] \arrow[d, "f_*"] & H^2(X) \\
H_2(\mathbb{C}P^2) \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\cong" rotated] & H^2(\mathbb{C}P^2) \arrow[u, "f^*"'] \arrow[d, phantom, "\cong" rotated] \\
\underset{\text{gen by } [\mathbb{C}P^1]}{\mathbb{Z}} & \underset{\text{gen by P.D.} [\mathbb{C}P^1]}{\mathbb{Z}}
\end{tikzcd}$$

so $\text{P.D.}(a) = f^*(\text{P.D.}[\mathbb{C}P^1]) = \text{P.D.}[f^{-1}(\mathbb{C}P^1)]$

i.e. $a = [\Sigma]$

<u>Big Question</u>: given $a \in H_2(X)$ what is the minimal genus of a surface $\Sigma \subset X$ such that $[\Sigma] = a$?

given $[\Sigma], [\Sigma'] \in H_2(X^4)$

define $[\Sigma] \cdot [\Sigma']$ = signed count of points in $\Sigma \cap \Sigma'$
(after they are made transverse)

<u>exercise</u>: $[\Sigma] \cdot [\Sigma'] = \langle P.D.([\Sigma]) \cup P.D.([\Sigma']), [X] \rangle$

so the "intersection pairing"

$$H_2(X) \times H_2(X) \to \mathbb{Z}$$

and cup product pairing

$$H^2(X) \times H^2(X) \to \mathbb{Z}$$

are "dual"

in particular, by Poincaré Duality, it is <u>non-degenerate</u>

it is also <u>symmetric</u> and <u>bilinear</u>

denote it $Q_X : H_2(X) \times H_2(X) \to \mathbb{Z}$

<u>lemma 8</u>:

If X is a 4-manifold made with only 0,2-handles (can also have 4-handle) and A is the linking matrix of the attaching circles of the 2-handles, then

$$H_2(X) \times H_2(X) \to \mathbb{Z}$$

is given by A in the basis A_i from lemma 6

Proof: recall $lk(L_1, L_j) =$ signed count $L_i \cap \tilde{\Sigma}_j$

Seifert surface in $S^3 = \partial B^4$

but this $=$ signed count $\Sigma_i \cap \tilde{\Sigma}_j$

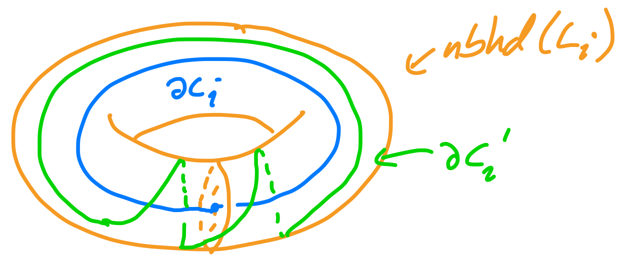
Seifert surface for L_i with interior pushed into B^4

$= $ signed count $\Sigma_i \cap \Sigma_j$

so $A_i \cap A_j = (C_i \cup \Sigma_i) \cap (C_j \cup \Sigma_j)$

$= \Sigma_i \cap \Sigma_j = lk(L_i, L_j)$

now for $A_i \cap A_i$ take $(C_i \cup \Sigma_i)$ and $C_i' =$ parallel to C_i



since these link n_i times the surface Σ_i' that glues to C_i' intersects Σ_i, n_i times

$Q_X : H_2(X) \times H_2(X) \to \mathbb{Z}$ is an invariant of X

in general, invariants of non-degenerate symmetric bilinear forms $Q : \mathbb{Z}^r \times \mathbb{Z}^r \to \mathbb{Z}$ are

1) rank$(Q) = r$

2) type — even if $Q(v,v)$ even $\forall v \in \mathbb{Z}^r$; odd otherwise

3) signature

$$\sigma(Q) = b_+ - b_-$$

$b_\pm$ = number of $\pm$ eigenvalues

4) definiteness

Q is positive definite if $b_- = 0$
negative definite if $b_+ = 0$
indefinite if $b_+ \neq 0 \neq b_-$

Algebraic Facts:

1) Q even then $\sigma(Q)$ divisible by 8

2) Q odd indefinite $\Rightarrow Q \cong \oplus^p (1) \oplus^q (-1)$ $\quad p,q>0$

3) Q even indefinite $\Rightarrow Q \cong \pm \oplus^p E_8 \oplus^q \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad q>0$

4) Q even definite $\Rightarrow$

$\sigma(Q)$	8	16	24	32	40	...
# of Q	1	2	24	7107	$>10^{51}$	...

lots!

5) Q odd definite $\Rightarrow$ even $\oplus \left(\pm \oplus^p (1)\right)$

Geometric Facts:

1) $X^4 = X_1 \cup_\partial X_2$ then $\sigma(X) = \sigma(X_1) + \sigma(X_2)$ $\quad \sigma(X) := \sigma(Q_X)$

(note X_i not closed, can still define Q_{X_i} it is just not non-degenerate but still has signature)

2) $\sigma(X_1 \# X_2) = \sigma(X_1) + \sigma(X_2)$

(clear from 1))

3) $\sigma(-X) = -\sigma(X)$ (easy)

reverse orientation

4) X closed oriented 4-manifold then

$$X = \partial W^5 \iff \sigma(X) = 0$$

5) X closed, oriented, smooth 4-manifold with $\pi_1 = 1$ and Q_X even, then $\sigma(X)$ divisible by 16

(Rokhlin's Th$^{\underline{m}}$ 1952)

6) If X is closed, oriented, smooth 4-manifold with $\pi_1 = 1$ and Q_X is definite, then $Q_X = \pm \oplus_k (1)$

(Donaldson 1983)

so the zoo of definite forms can be ignored when studying smooth 4-manifolds!

<u>Fact</u>: Every closed orientable 3-manifold bounds a 4-manifold $X = 0\text{-handle} \cup 2\text{-handles}$ where the framings are even

(proof is just Kirby calculus, but a bit long)

now let M be a homology 3-sphere

(i.e. $H_*(M) \cong H_*(S^3)$)

let X be a 4-manifold as in fact above with $\partial X = M$

since M a homology sphere Q_X is non-degenerate

$\therefore$ Alg. fact 1) $\Rightarrow \sigma(Q_X)$ is divisible by 8 set $\mu(M) = \sigma(X)/8 \mod 2$

lemma 9:

$\boxed{\mu(M) \text{ is well-defined}}$

Proof: need to see $\mu(M)$ is independent of X

so let X, X' be 2 such 4-manifolds

i.e. $Q_X, Q_{X'}$ even, simply connected,

and $\partial X = M = \partial X'$

let $W = X \cup_M -X'$

W is a closed smooth 4-manifold with $\pi_1(W) = 1$

and Q_W is even

$\therefore$ Rokhlin (Geom. fact 5)) $\Rightarrow \sigma(Q_M)$ divisable by 16

that is $\sigma(Y) \underset{\substack{\uparrow \\ \text{geom fact 1)}}}{=} \sigma(X) + \sigma(-X') \underset{\substack{\uparrow \\ \text{geom fact 3)}}}{=} \sigma(X) - \sigma(X') \equiv 0 \mod 16$

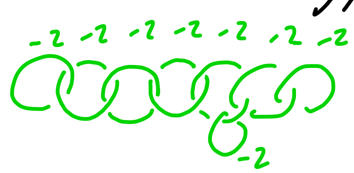
so $\sigma(X) \equiv \sigma(X') \mod 16$

and $\sigma(X)/8 \equiv \sigma(X')/8 \mod 2$

$\mu(M)$ is called the Rokhlin invariant of M

example:

1) P = Poincaré homology sphere

= 

$\leftarrow \sigma = -8$

so $\mu(P) = 1$

exercise: P is also  -1

note: this implies P does not bound a homology 3-ball! Determining when a homology 3-sphere bounds a homology ball is a major area of study

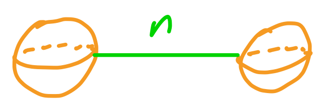
2) $\mu(S^3) = 0$

C. Moves between Kirby diagrams

just like in dimensions 2 and 3 we can do the following manipulations to Kirby diagrams in dimension 4

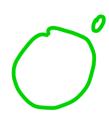
① 1/2-cancelling pairs

add or delete



② 2/3-cancelling pairs

add or delete

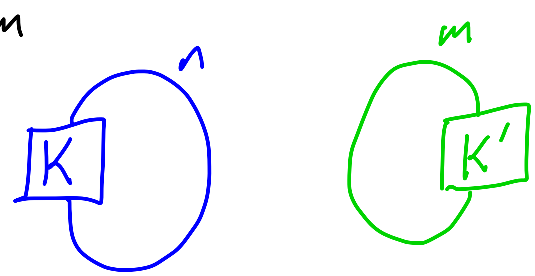


$\cup$ 3-handle

exercise: "see" the S^2 that the 3-handle is attached to

③ 2-handle slides

given



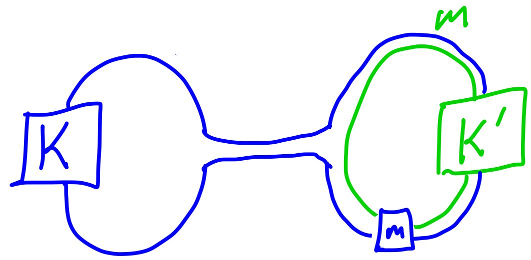
(K, K' might link and go over 1-handles ...)

we know we can take a copy of K' using the framing (coming from $\partial D^2 \times \{p\} \subset \partial D^2 \times D^2$ where $p \in \partial D^2$)

and push K over it (i.e. isotop over

$D^2 \times \{p\} \subset D^2 \times \partial D^2 = \partial_+ h^2)$

this gives



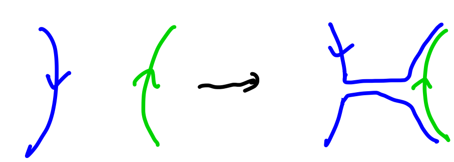
but what is the framing on K after the slide?

here we assume handles do not go over 1-handles so they represent 2-homology classes

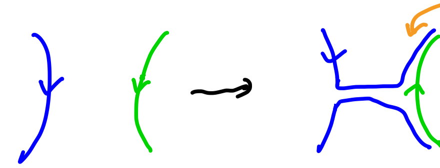
well recall, the handle attached to K gives a 2-homology class represented by a surface Σ_K (actually need to orient K for this, so that Σ_K oriented)

and K' gives a homology class $\Sigma_{K'}$ (after orienting)

exercise: Show that the homology class represented by the handle after the slide is $\Sigma_K + \Sigma_{K'}$ if orientations agree

i.e. 

or is $\Sigma_K - \Sigma_{K'}$ if they disagree

i.e. 

orientation on strand changed!

hint: the difference between these classes is an embedded 3-manifold with boundary
orient the manifold and it gives a 3-chain whose boundary gives the homology between the classes

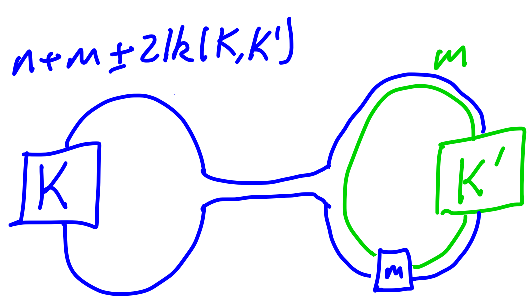
now if h_K and $h_{K'}$ are the homology classes associated to the handles attached to K and K'

then the knot after the slide represents the homology class $h_K \pm h_{K'}$

now from lemma 8 we know the framing on the handle after the slide is

$$(h_K \pm h_{K'})^2 = h_K \cdot h_K \pm 2 h_K \cdot h_{K'} + h_{K'} \cdot h_{K'}$$
$$= n + m \pm 2 \text{ linking}(K, K')$$

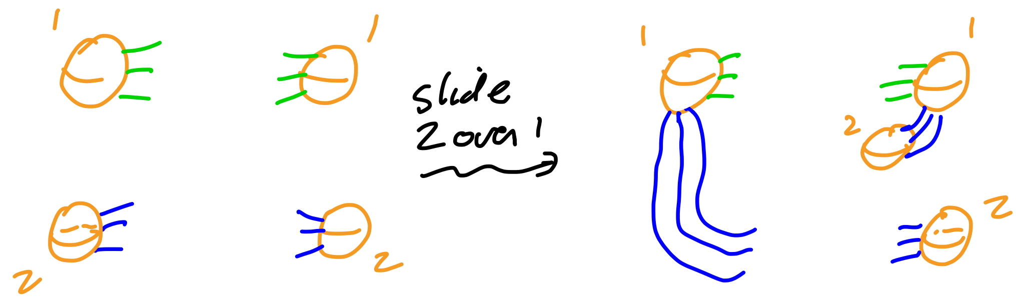
so the framing after the slide is



exercise: with the conventions set up for framings if K, K' go over 1-handles show the formula still works.

④ 1-handle slides

one can usually avoid these, but they are "easy"



exercise: check framing on the blue

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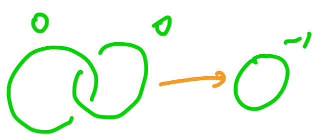
example:

$$(S^2 \times S^2) \# \overline{\mathbb{C}P}^2 \cong \mathbb{C}P^2 \# \overline{\mathbb{C}P}^2 \# \overline{\mathbb{C}P}^2$$


$\cup$ 4-h

$\cong$ handle slide


$\cup$ 4-h

$\cong$ handle slide


$\cup$ 4-h

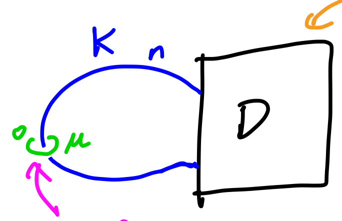
$\cong$ handle slide


$\cup$ 4-h

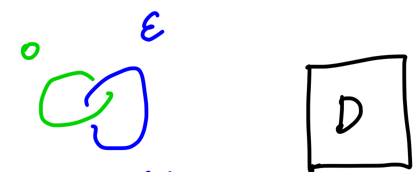
we first observe a very useful handle slide (when you can do it)

lemma 10:

if you have a diagram with only 2-handles and you see



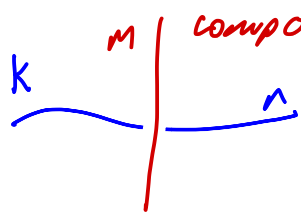
then after handle slides you can get to



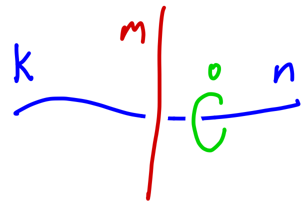
$\varepsilon = \begin{cases} 1 & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$

Proof: we first see the blue knot K can be unlinked from D

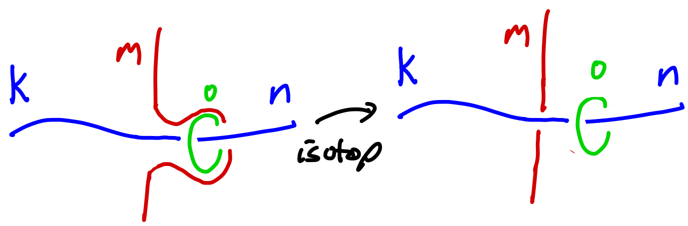
note: if you see a crossing between K and a component of D:



then you can isotop μ to



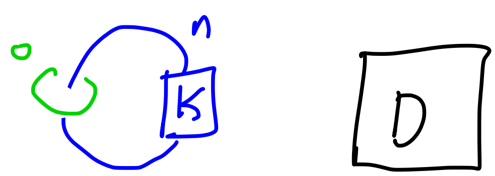
now slide red over μ to get



i.e. you can chang any crossing of K with components of D

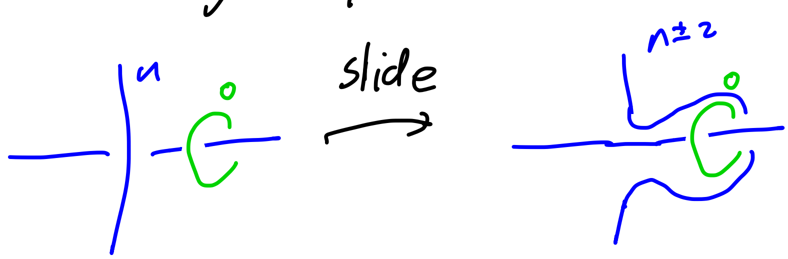
exercise: using this you can isotop $K \cup \mu$ to be disjoint from D

i.e.

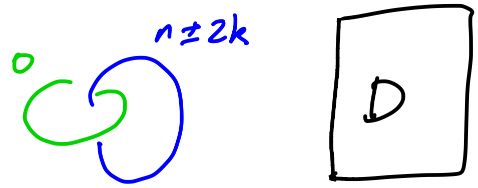


now you can also change crossings of K with itself, but when you do, the framing on K changes by ± 2

eg.

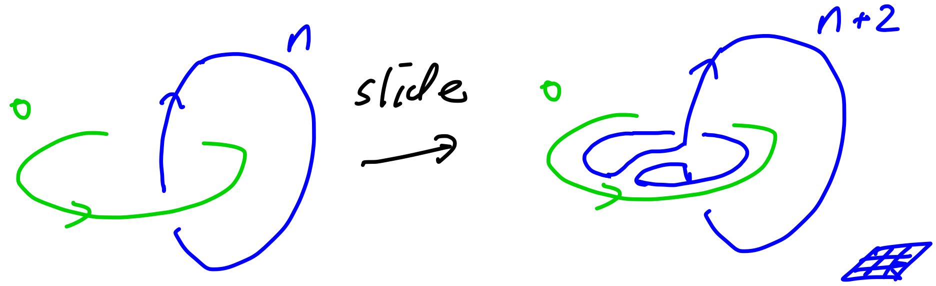


so we get to

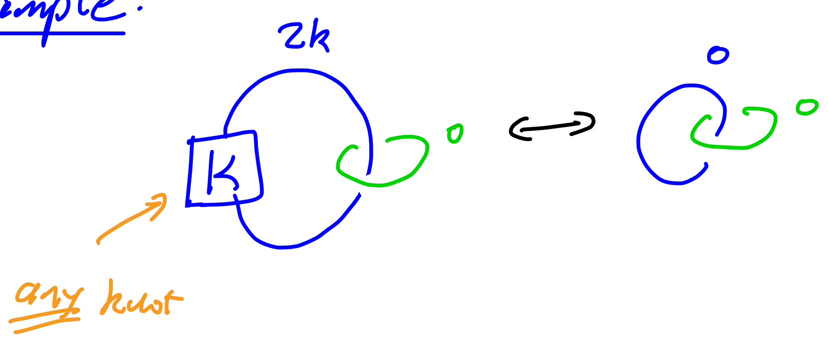


now notice you can do further slides of K over μ to chang framing by any even #

e.g.



example:



$= (S^2 \times S^2) -$ ball

Construction: Doubling

given a manifold X with $\partial X \neq \emptyset$

the double of X is

$$D(X) = X \amalg \overline{X} \Big/ \text{identify } \partial X \text{ with } \partial \overline{X} \text{ by the identity}$$

$\overline{X}$ — X with reversed orientation

$$\cong \partial([0,1] \times X)$$

$$= (X \times \{0\}) \cup (\partial X \times [0,1]) \cup (X \times \{1\})$$

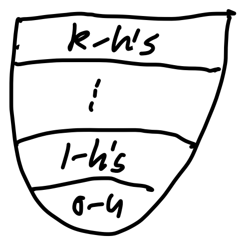
s11

$\overline{X} \cup X/\sim$

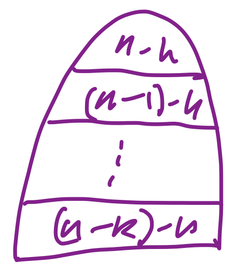
exercise: if X has boundary, Y does not, then $D(X \times Y) = D(X) \times Y$

So how do we double a handle body?

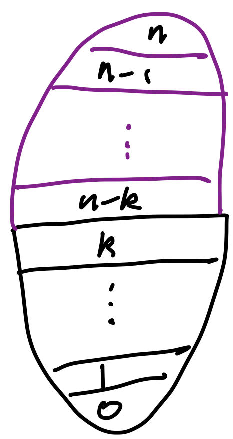
given



take an upside down copy



and glue

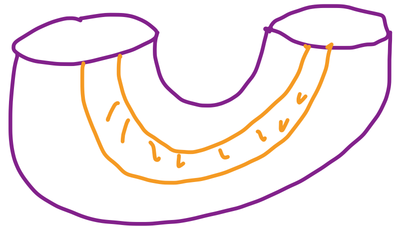


you can think about putting a mirror on top of the first handlebody and looking at the reflection of the handles

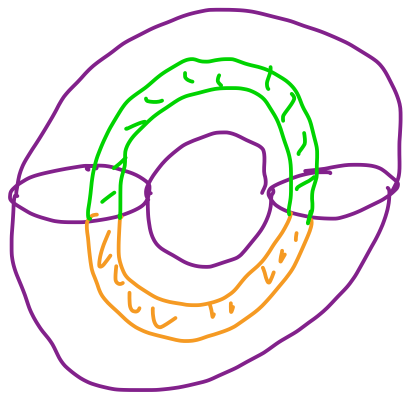
example:

given

we draw so ∂ at top

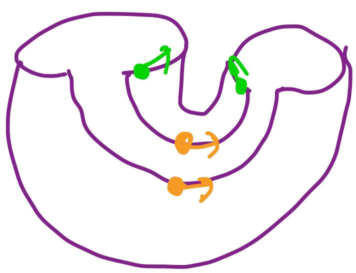


double



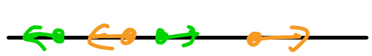
note the reflection of the orange is a green 1-handle attached to the <u>belt sphere</u> of orange 1-handle

so we can push this into ∂(0-handle) and see our handles are attached



so 

double



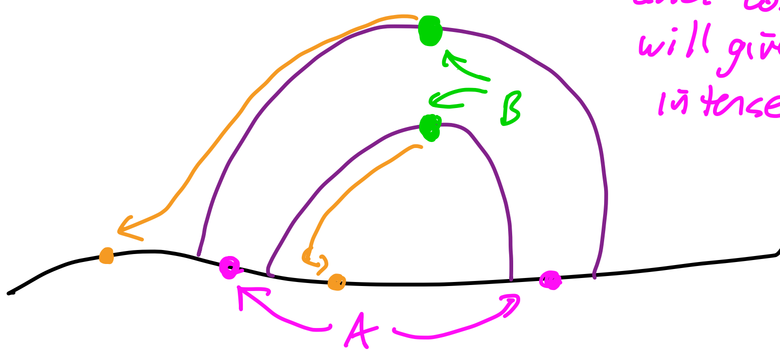
We now consider a 4-dimensional manifold with only 0, 1, 2-handles

the 0, 1-handles become 3, 4-handles which we do not need to draw

so only consenned with 2-handles

as before we the doubled 2-handle will be a 2-handle attached to the belt sphere with framing 0

since cocore of orig. handle and core of new handle will give an S^2 with self-intersection 0

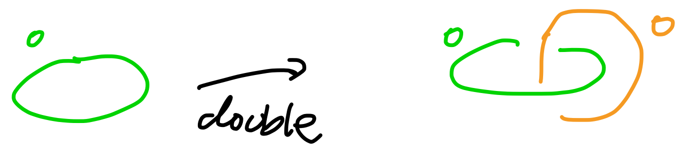


exercise: check this

as discussed before when belt sphere isotoped into $\partial(0\text{-}h)$ it will be a meridian for the original attaching sphere

examples:

1) $D(S^2 \times D^2) = S^2 \times D(D^2) = S^2 \times S^2$

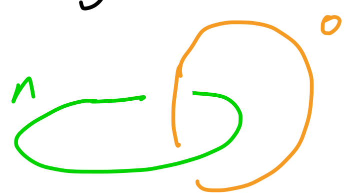


$\cup$ 4-handle

2) more generally the double of the D^2 bundle E_n over S^2 with Euler number n is an S^2-bundle over S^2

exercise: show this

the Kirby diagram is

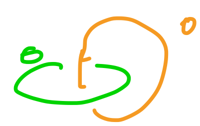


$\cup$ 4-handle

exercise:

"see" the orange 2-handle the fiber of the S^2-bundle and the green gives a section

If n is even the above (by lemma 10) is diffeo to



$\cup$ 4-handle

and if n odd

$\cup$ 4-handle

exercise:

1) Show there are only two S^2-bundles over S^2

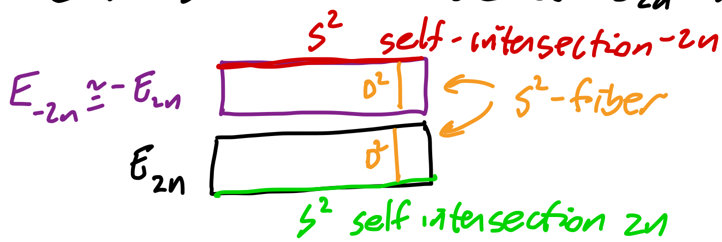
2) show (without Kirby calculus) that $S^2 \times S^2$ contains two disjoint spheres, one with self-intersection $2n$ the other with self-int $-2n$

maybe also do this with Kirby calculus

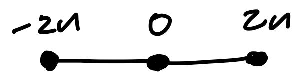
e.g. in  ∪ 4-handle

where is the disjoint $-2n$ sphere?

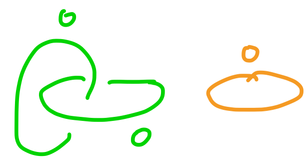
note: since $S^2 \times S^2$ is the double of E_{2n} we see



so in $S^2 \times S^2$ we see the plumbing



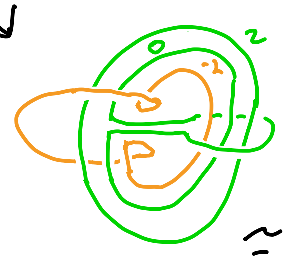
where is it?

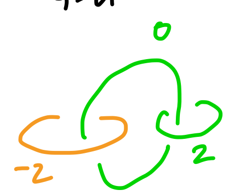
 ∪ 4-h $\xrightarrow{\text{add cancelling 2/3 pair}}$  ∪ 3-h 4-h

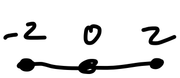
slide

 ∪ 3-h 4-h

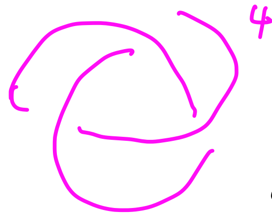
slide 2 horizontal curves over vertical one (add one subtract other

 ∪ 3-h 4-h

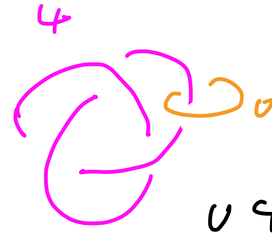
$\cong$  ∪ 3-h 4-h

note: the complement of  in $S^2 \times S^2$ is $S^1 \times D^3$! (0 & 1-handle)

3)

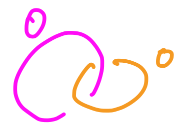


$\xrightarrow{\text{double}}$



$\cup$ 4-4

from lemma 10 this is



$\cup$ 4-4

so $S^2 \times S^2$

lemma 11:

> any compact oriented 3-manifold embeds is $\#_k S^2 \times S^2$ for some k

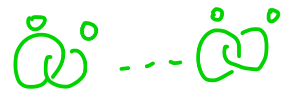
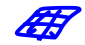
Remark: Bizaca and E. used this to build infinitely many smooth structures on $M \times \mathbb{R}$ for any compact, orientable 3-manifold M.

Proof: we noted above any 3-manifold M is the boundary of a 4-manifold X made with one 0-handle and some number of 2-handles attached to $K_1 \dots K_k$ with even framing

now $M \hookrightarrow D(X)$ but a Kirby diagram for $D(K)$

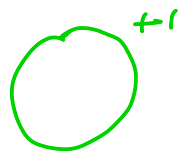
is $K_1 \cup \mu_1 \dots K_k \cup \mu_k$ wher μ_i are meridians for K_i and have 0-framing

lemma 10 $\Rightarrow$ can isotop so all K_i are unknots
have 0 framing and are unlinked

i.e.  $\cup$ 4-h $= \#_k S^2 \times S^2$ 

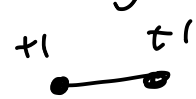
let's use the above to explore surfaces, and their complement, in $\mathbb{C}P^2$

of course $\mathbb{C}P^2$ is

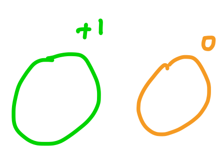
 $\cup$ 4-handle

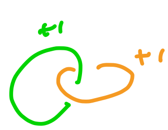
so we see an S^2 with self-intersection 1
and a complement of its nbhd
is B^4

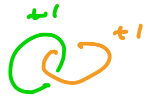
if we take this sphere and push off a second copy
we see the plumbing



in $\mathbb{C}P^2$. How can we "see" this in
the Kirby diagram?

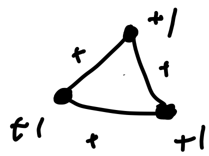
 $\cup$ 4-handle $\xrightarrow[\text{2/3 pair}]{\text{add}}$  $\cup$ 3-handle
4-handle

$\xrightarrow[\text{slide}]{\text{handle}}$  $\cup$ 3-handle
4-handle

now we see  $= \overset{+1}{\bullet}\!\!-\!\!\overset{+1}{\bullet}$

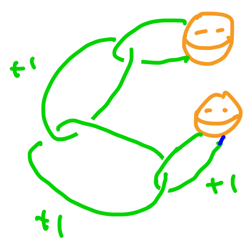
and the complement is $S^1 \times D^3 = 0\text{-h} \cup 1\text{-h}$

now let's try



↙ just 3 copies of $\mathbb{C}P^1 \subset \mathbb{C}P^2$

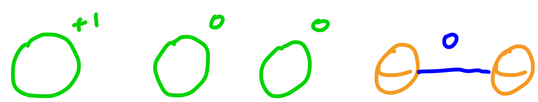
from above we know a Kirby diagram for this is



now $\mathbb{C}P^2$ is

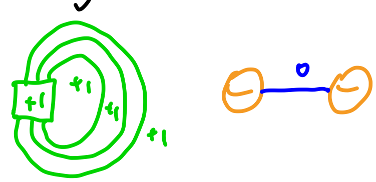
$\cup$ 4-handle

add 2 pairs of cancelling 2/3 pairs and a cancelling 1/2 pair



$\cup$ 2 3-handles
4-handle

slide 2 green 0-framed handles over +1

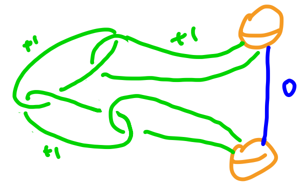


$\cup$ 2 3-handles
4-handle

isotop

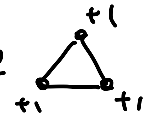
$\cup$ 2 3-handles
4-handle

slide one of the +1 framed handles over blue ×2



∪ 2 3-handles
4-handle

so we see



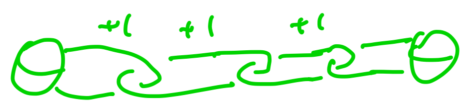
in $\mathbb{C}P^2$!

what's it's complement?

Well it is the blue 2-handle ∪ 2 3-handles
∪ 4-handle

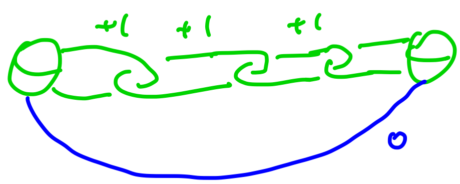
to see this we turn these handles upside down!

so we think of them as attached to boundary of the plumbing which is the 3-mfd



now green means 3-manifold and blue means 4D 2-handles

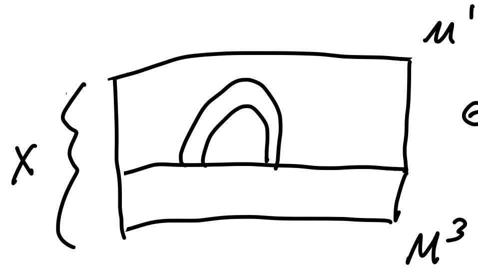
so complement is



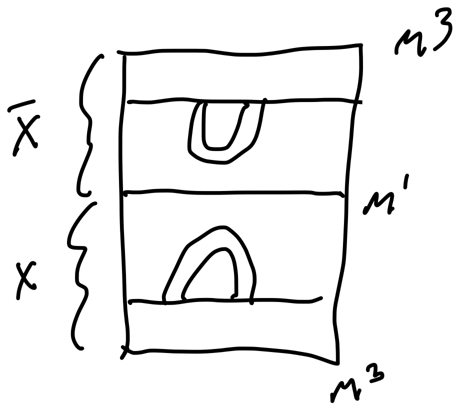
∪ 2 3-handles
4-handle

we know the 3 and 4-handles upside down are

we need to see how upsidedown blue handle is glued on for this we do a "relative double"



take copy of this and glue upside down

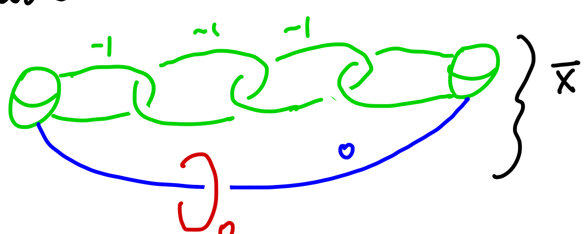


we do this just like in the normal double

but to get X upside down need to start with $\bar{X}$

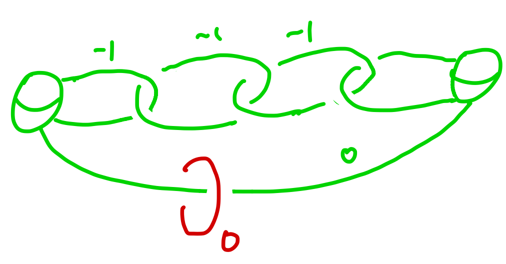
<u>exercise:</u> to get $\bar{X}$ from X just mirror the knots and negate framings

so we have



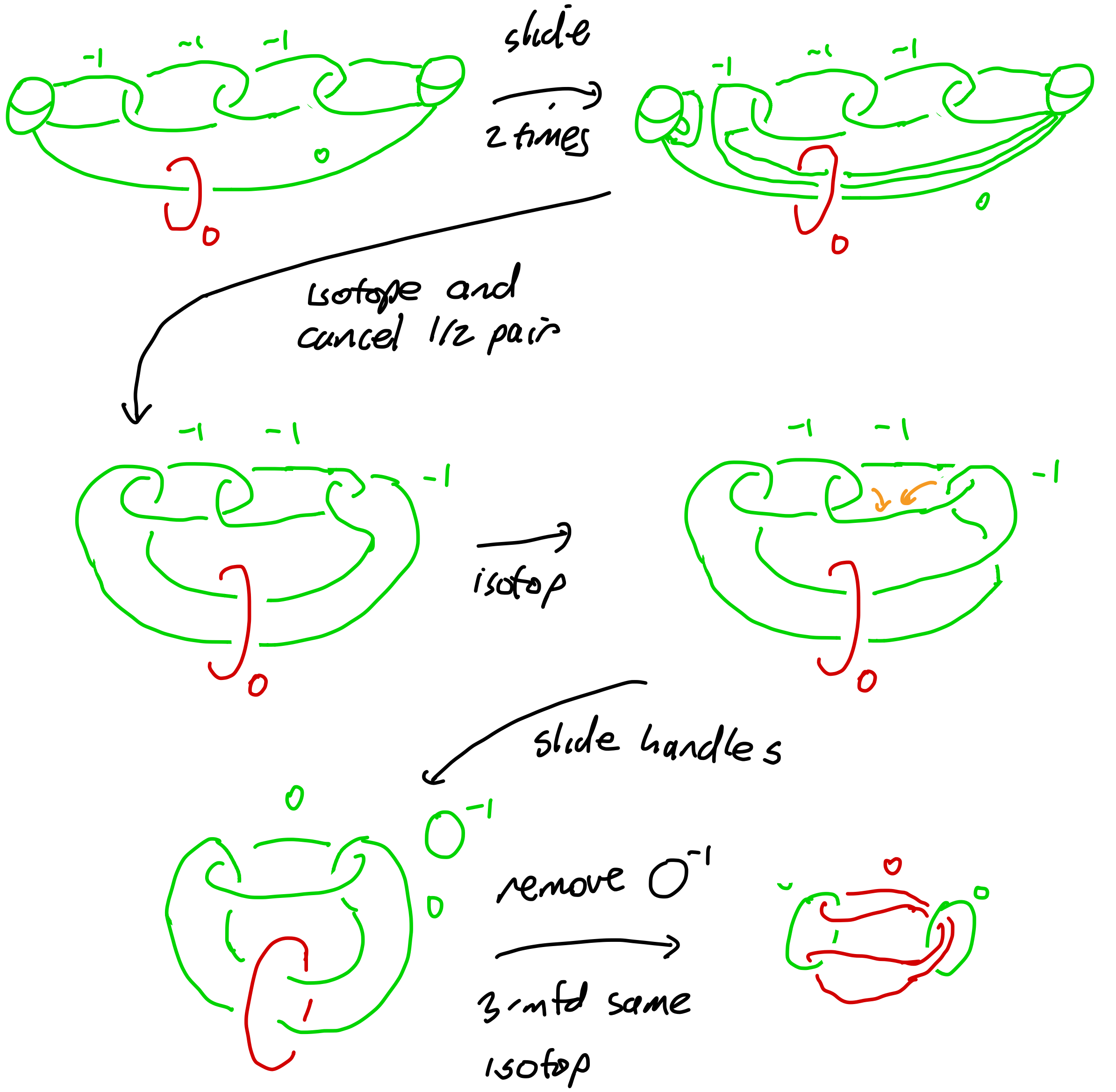
↖ double blue 2-handle

so upside down X is



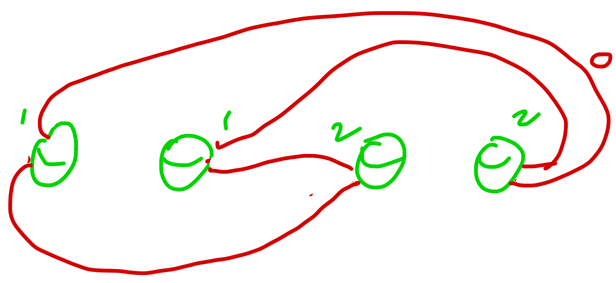
we know you can glue 2 3-handles and a 4-handle to the green 3-manifold so it <u>is</u> $\#_2 S^1\times S^2$

let's manipulate the diagram untill we see (and drag the red handle along)

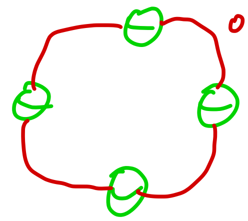


note:

but this is the upside down 3 and 4 handles so the complement of the plumbing is



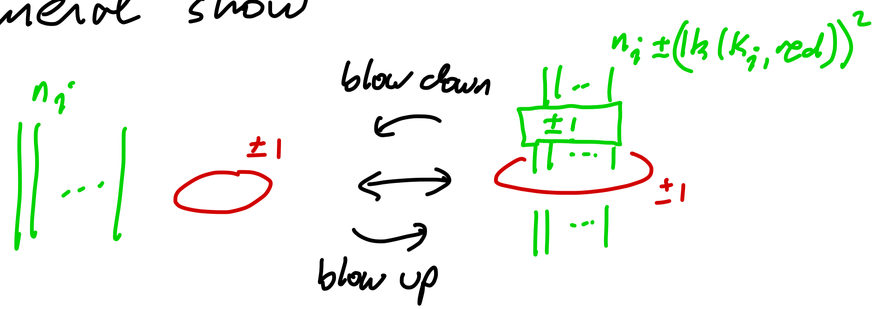
isotop to



so complement of plumbing is nbhd of T^2!

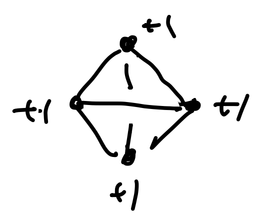
exercise: above we used a useful move

in general show



exercise: can also find a "complete graph on n vertices" with each vertex labled $+1$ in $\mathbb{C}P^2$

e.g.



see this in Kirby diagram and determine complement.

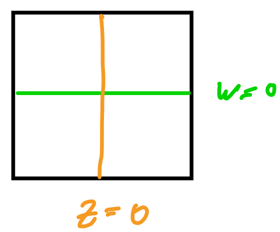
note: if you see $\overset{n}{\bullet}\!\!-\!\!\!-\!\!\overset{m}{\bullet}$ ← call S_1, S_2 orient so $\cap$ is $+$

then you can create $\overset{n+m+2}{\bullet}$ or $\overset{n+m-2}{\bullet}$

in a nbhd of $\underset{n}{\bullet}\!\!-\!\!\!-\!\!\underset{m}{\bullet}$

to see this notice that a nbhd of the intersection point is

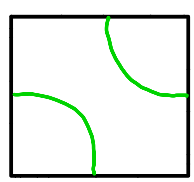
$$\begin{array}{ccc} D^4 = & D^2 \times D^2 & \subset \mathbb{C}^2 \\ & \cup \quad\;\; \cup & \\ & D^2\times\{0\} \quad D^2\times\{0\} & \\ & \| \qquad\qquad \| & \\ & \{z=0\} \quad \{w=0\} & \end{array}$$



so surfaces near intersection are

$$zw = 0$$

consider $zw = \varepsilon \qquad \varepsilon > 0$



this is an annulus A +

$$\phi : S^1 \times (0,\infty) \to \mathbb{C}^2$$
$$(\theta, r) \mapsto \varepsilon(re^{i\theta}, \tfrac{1}{r}e^{-i\theta})$$

we can replace the 2 intersecting disks by this annulus to get a surface $S_1 \# S_2$

exercise:

$S_1 \# S_2$ is homologous to $S_1 \cup S_2$

hint: region between them is a 3-manifold with boundary

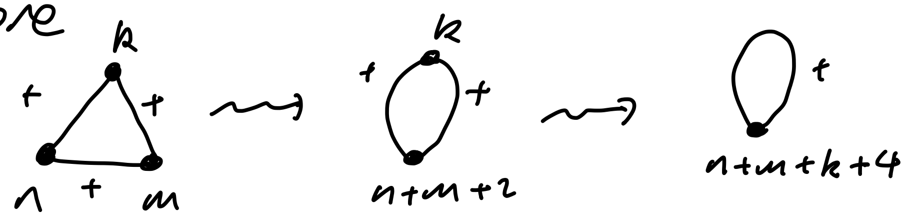
so $S_1 \# S_2$ is an S^2 with self intersection

$$[S_1 \cup S_2]^2 = ([S_1]+[S_2])^2$$
$$= [S_1]^2 + 2[S_1]\cdot[S_2] + [S_2]^2$$
$$= n + 2 + m \quad \text{done!}$$

you can use $A_- = \{zw = -\varepsilon\}$ to get $S_1 \# -S_2$ with self-int. $n+m-2$.

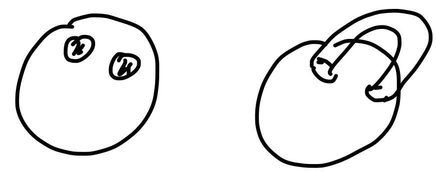
in general

even more



these are just as above

but now when you "resolve" last double point, you take "internal" connect sum

i.e. 

i.e. get a torus!

so we get $\overset{\bullet}{(1, n+m+k+6)}$

exercise:

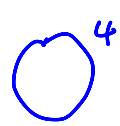
show the homology class $n \in \mathbb{Z} \cong H_2(\mathbb{C}P^2)$ (where $[\mathbb{C}P]$ generates H_2)

for n positive is represented by

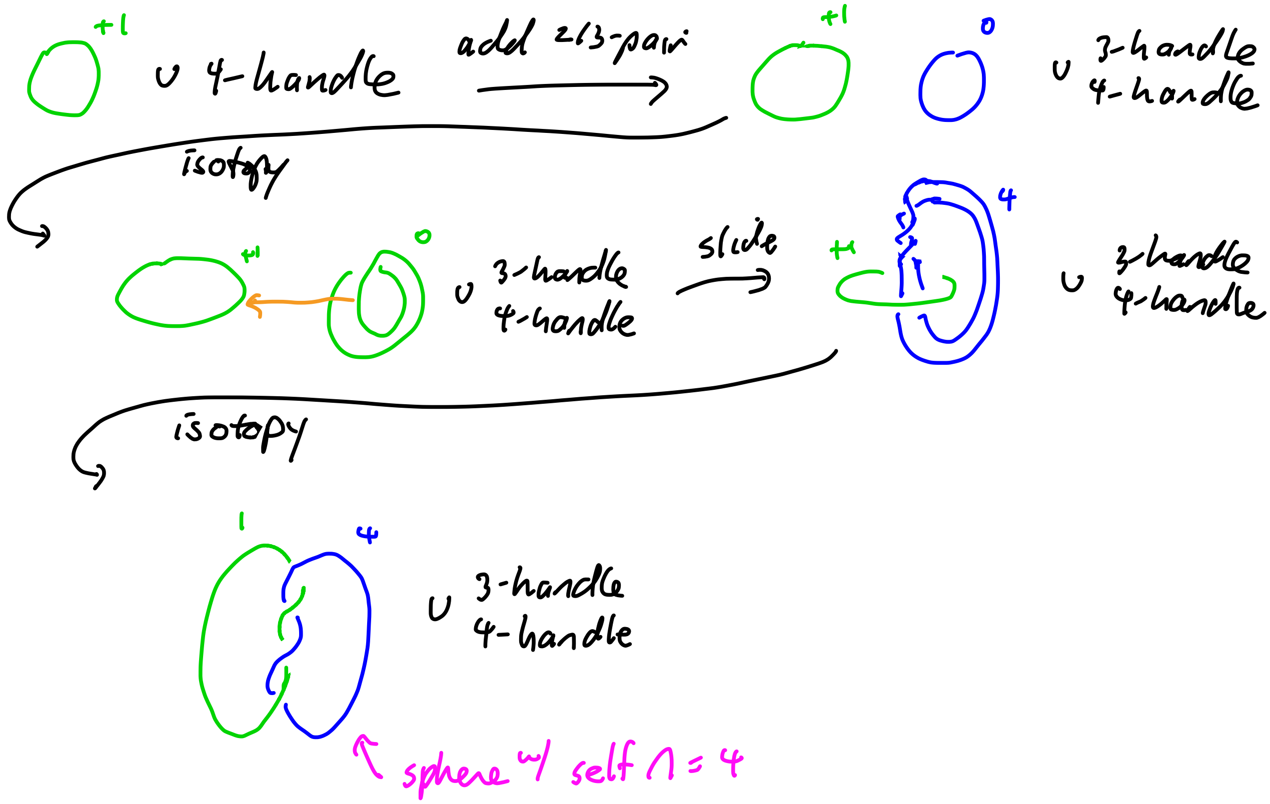
a surface of genus $\frac{(n-1)(n-2)}{2}$ with self-intersection n^2

(e.g. from above $\overset{+1}{\bullet}\!\!-\!\!\overset{+1}{\bullet}$ has $\overset{4}{\bullet}$ in its nbhd)

so $\exists$ a sphere of self intersection 4 in $\mathbb{C}P^2$

let's "see" 

in $\mathbb{C}P^2$



What is its complement?

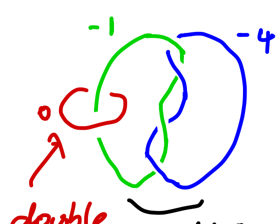
it is made up of a 2-h, 3-h, 4-h

the 4,3-handles upside down is 

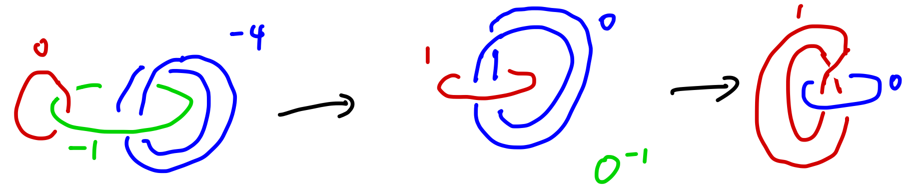
we need to see where the 2-handle goes

as we did above we think of  as
representing a 3-manifold and
the green 2-handle is attached to it

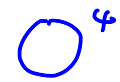
we turn this upsidedown
(need to reverse orientation)

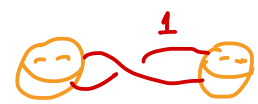


this describes $S^1 \times S^2$ (so can glue 3,4-handle)



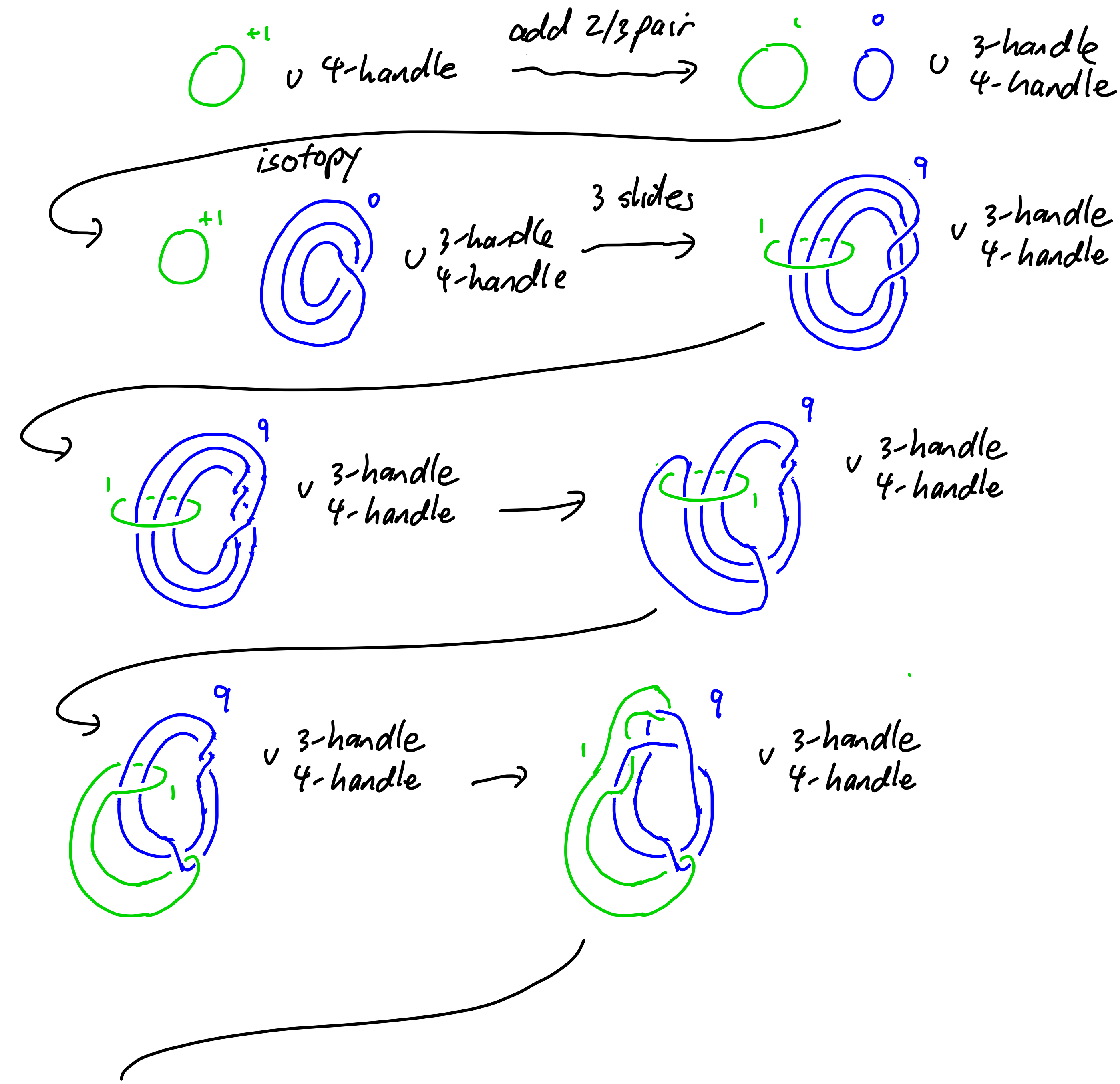
the blue is $S^1 \times S^2$ can think of as ∂(0-,1-handle)

so complement of  is



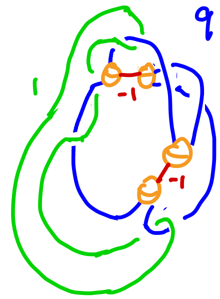
now we know a torus with self $\cap$ 9

let's see it

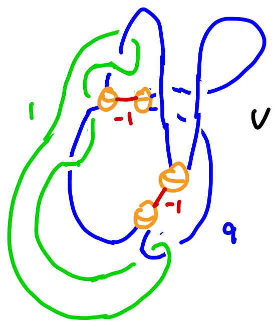


$\cup$ 3-handle
4-handle

$\xrightarrow{\text{add 1-2 pairs}}$

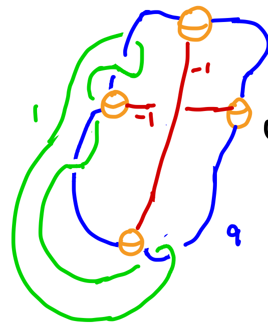


$\cup$ 3-handle
4-handle

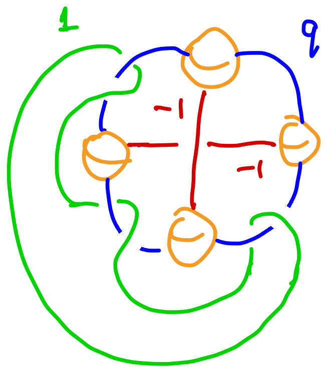


$\cup$ 3-handle
4-handle

$\longrightarrow$

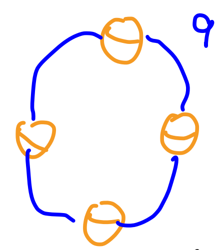


$\cup$ 3-handle
4-handle



$\cup$ 3-handle
4-handle

so we see

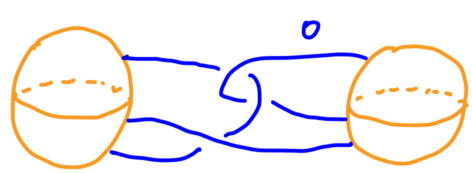


in $\mathbb{C}P^2$

<u>exercise:</u> find the complement of $\bullet^{(1,9)}$ in $\mathbb{C}P^2$

<u>example:</u>

the Mazur manifold is

$M =$ 

<u>exercise:</u> Show $\pi_1(M) = \{e\}$

and $H_*(M) \cong H_*(B^4)$

so M is contractible (ie. $\simeq B^4$) homotopy equivalent

is M homeomorphic to B^4?

No

exercise: 1) ∂M is not simply connected

2) let $\Sigma(2,5,7) = \{ z_1^2 + z_2^5 + z_3^7 = 0 \} \cap S^5$ unit sphere in $\mathbb{C}^3$

show $\partial M = \Sigma(2,5,7)$

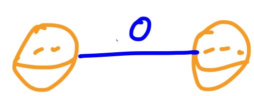
(need to find a surgery diagram for $\Sigma(2,5,7)$)

so in 4D $\simeq B^4$ don't have to be B^4

(in 2D, 3D, $\simeq B^n \Rightarrow \cong B^n$)

note: $M \times [0,1]$ is a 5-manifold made with a 0, 1, 2-handle the 2-handle is attached to a curve in $S^3 \subset S^4 = \partial B^5$ (equator)

since "crossings" in S^4 can be changed

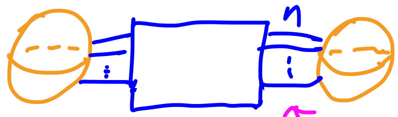
$M \times [0,1] \cong$  $\cong B^5$

so $M \times [0,1] = B^4 \times [0,1]$! (terms in product not uniquely determined)

also $D(M) = \partial(M \times [0,1]) = S^4$

$\therefore \Sigma(2,5,7) = \partial M$ embeds in S^4!

a generalized Mazur is anything of the form



goes over 1-handle algebraically 1 time

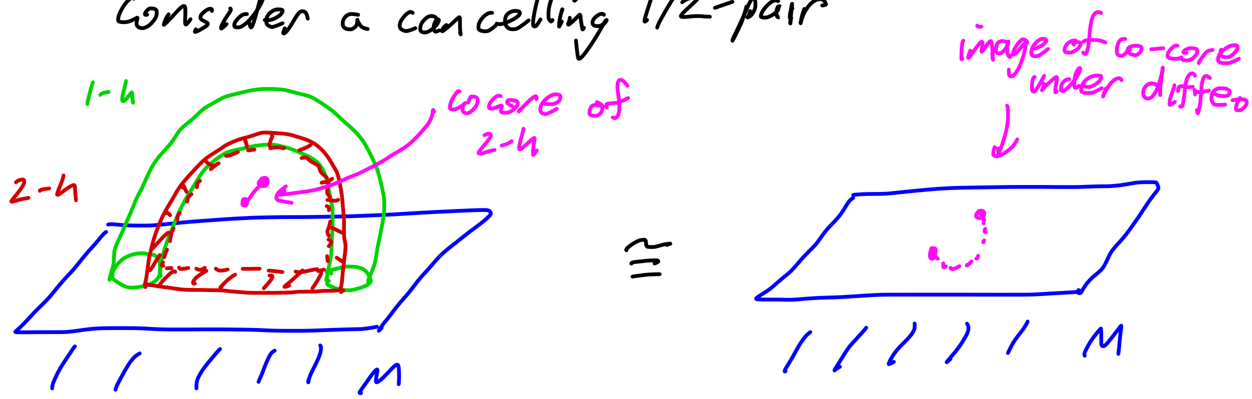
any such manifolds has same properties as above

<u>Fun Fact:</u> if M is a generalized Mazur manifold and ∂M is a Heegaard Floer L-space then $M \cong B^4$ (Conway-Tosun)

D. 1-handles again

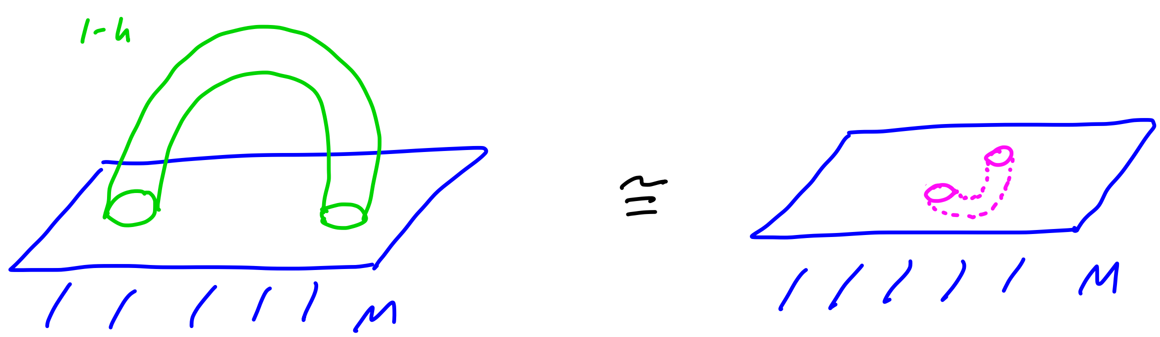
let's start in 3D:

consider a cancelling 1/2-pair



removing a nbhd of the cocore is the same as removing the 2-handle

∴ same as adding the 1-handle



note: the arc (= co-core) on the left is isotopic to an arc in ∂M connecting end points

easy to see if you

1) take any arc α in ∂M
2) push its interior into int M
3) remove a neighborhood of the arc

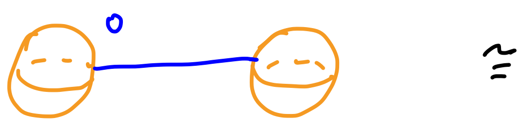
they you have attached a 1-handle to M

exercise:

1) check this
2) show a 2-handle "goes over" the 1-handle if it crosses α in ∂M

now for 4D:

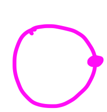
again consider a cancelling 1/2 pair



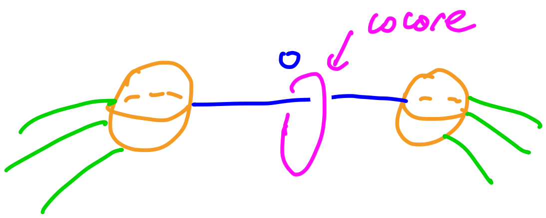
the boundary of the cocore of the 2-handle (as we have seen above) is

removing a nbhd of the cocore removes the 2-handle and leaves the 1-handle

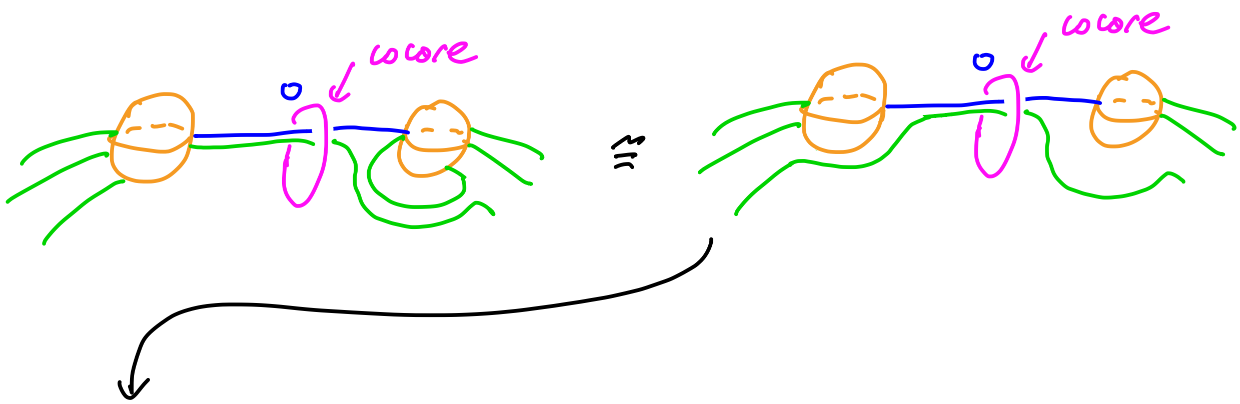
take disk this unknot bounds and push it into interior of 0-handle, then remove a nbhd of it

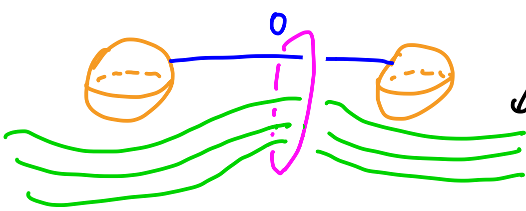
denote this by 

now what happens if 2-handles run over 1-handle



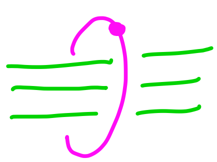
slide green over blue





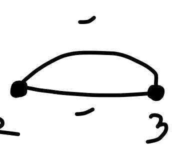
framings unchanged

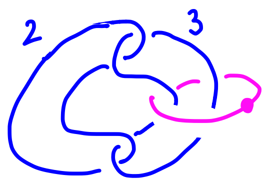
≅



examples:

1)





2) T^3